Kac's solution of the telegrapher's equation for tunneling time analysis: An application of the wavelet formalism

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A plausible description of traversal time was given, both in classically allowed and forbidden regions, through a path-integral solution of the telegrapher's equation. This analysis was applied to a simulation based on microwave propagation in a waveguide considered as a one-dimensional system. An extension of the analysis has been performed in order to compare the traversal (or delay) time results relative to a beat-envelope signal with those as deduced from the distribution function of the randomized time and its analytical continuation in imaginary time. Subsequently, in tight analogy with a step signal in an electronic circuit (zero-dimensional system), we have searched for a simulation of traversal processes in real time, even for classically forbidden (tunneling) processes. First we have considered a finite series expansion of harmonic functions of the signal in the neighborhood of its rise, and applied the above mentioned procedure to each harmonic, implying analytical continuation in imaginary time and an arbitrary truncation in the range of the signal. Then, in order to avoid these shortcomings, we have considered a waveletlike description of the signal.

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I. INTRODUCTION TO A STOCHASTIC ANALYSIS OF TUNNELING

In the lectures delivered in 1956 at the Magnolia Petroleum Company and Socony Mobil Oil Company, Kac made [1], as reported by DeWitt-Morette and Foong [2], some tantalizing remarks while presenting a path-integral solution of the telegrapher's equation. One of these remarks is that Kac's method can be thought of as "randomizing the time" in the solutions of such an equation without dissipation, and averaging over all the possible paths. Moreover, he stated that "this amusing observation persists for all equations of this form in any number of dimensions." Here, however, we focus attention on the one-dimensional wave equation in connection with the semiclassical analysis of traversal time—the time required for a particle to go from an initial to a final position—both in classically allowed and forbidden spatial regions. The latter are strictly connected to the tunneling processes for which the determination of the time duration is still an open problem. We shall see that a modeling based on the telegrapher's equation—the basic assumption being the analogy between (relativistic) particle motion and wave propagation—demonstrates further capabilities of the semiclassical approaches to interpret tunneling, making them more similar to pure quantum-mechanical approaches [3,4]. The wavelet analysis, as developed in connection with relativistic wave equations [5], appears a promising tool for further developments of the theory.

Kac's work basically consists of demonstrating that the telegrapher's equation is equivalent to a stochastic motion of a particle, moving on a straight line with constant velocity v, which suffers collisions that can reverse its velocity with probability $a\Delta t$, after each step Δx, and probability $1-a\Delta t$ of continuing in the same direction. More specifically, let us consider the telegrapher's equation in the form

$$\frac{1}{v^2}\frac{\partial^2 F}{\partial t^2}+\frac{2a}{v^2}\frac{\partial F}{\partial t}-\frac{\partial^2 F}{\partial x^2}=0\ , \qquad (1)$$

where a is a positive constant, $F(x,0)=\phi(x)$ is an "arbitrary" function such that $(\partial F/\partial t)_{t=0}=0$, and $\phi(x,t)$ is a solution of the wave equation (1) without dissipation ($a=0$). The solution of Eq. (1) can be put in the form of an average over all the possible paths

$$F(x,t)=\tfrac{1}{2}[\langle\phi(x+vS(t))\rangle+\langle\phi(x-vS(t))\rangle]\ , \qquad (2)$$

where $S(t)$ is the randomized time defined by

$$S(t)=\int_0^t(-1)^{N(\tau)}d\tau\ , \qquad (3)$$

$N(\tau)$ being a random variable with Poisson distribution of intensity a, that is, the probability to be $N(\tau)=k$ ($k=0,1,\ldots$) is

$$P(N(\tau)=k)=e^{-a\tau}\frac{(a\tau)^k}{k!}\ . \qquad (4)$$

The meaning of the randomized time can be seen by evaluating the first moment $\mu_1(t)\equiv\bar{\tau}$ of this quantity by the average

$$\bar{r}(t)=\left\langle \int_0^t (-1)^{N(\tau)} d\tau \right\rangle \quad (5)$$

or, by interchanging the average and the integration, by

$$\bar{r}(t)=\int_0^t \langle (-1)^{N(\tau)} \rangle d\tau \; . \quad (6)$$

Now, from the definition of the average and from Eq. (4) it follows that

$$\langle (-1)^{N(\tau)} \rangle = \sum_{k=0}^{\infty} (-1)^k e^{-a\tau} \frac{(a\tau)^k}{k!} = e^{-2a\tau} \quad (7)$$

and, by substituting into Eq. (6), we have immediately

$$\bar{r}(t)=\int_0^t e^{-2a\tau} d\tau = \frac{1}{2a}[1-e^{-2at}] \; . \quad (8)$$

The average time $\bar{r}$ has to be interpreted as the fictitious time it would take a particle to reach the average distance $X=v\bar{r}$ if it was always moving with the velocity v without reversal. So the result of Eq. (8) clearly accounts for the fact that dissipation continuously reduces the effective speed of the motion with the distance tending to the saturation value $v/2a$. By multiplying by v and inverting Eq. (8), we see that the average true time required to reach the distance $X=v\bar{r}$ is given by

$$t=-\frac{1}{2a}\ln\left[1-2a\frac{X}{v}\right] \; . \quad (9)$$

For $a \to 0$, Eq. (9) correctly gives the classical result $t=X/v$, while for the saturation value, $X=v/2a$, t tends toward infinity. More generally, it can be demonstrated that the solution $F(x,t)$ of Eq. (1) can be expressed by a quadrature

$$F(x,t)=\tfrac{1}{2}\int_{-\infty}^{\infty}[\phi(x,r)+\phi(x,-r)]g(t,r)dr$$
$$=\tfrac{1}{2}\int_0^{\infty}[\phi(x,r)+\phi(x,-r)]h(t,r)dr \; , \quad (10)$$

where $g(t,r)$ is the distribution of $S(t)$, while $h(t,r)\equiv g(t,r)+g(t,-r)$ is the distribution of $|S(t)|$. The functions $h(t,r)$ and $g(t,r)$ were evaluated by a Laplace-transform analysis and they are given in Refs. [2] and [6], respectively.

By using the distribution function $g(t,r)$, the result of Eq. (8) can be obtained as

$$\bar{r}(t)=\int_{-\infty}^{\infty} r\, g(t,r)dr = \frac{1}{2a}(1-e^{-2at}) \quad (11)$$

and also turns out to be identical to the average of the first passage time $\bar{r}_1(t)$ [6,7].

Returning to Eq. (10), the interest of that result lies in the fact that if we know a solution $\phi(x,t)$ of the wave equation without dissipation, we can obtain the solution of the complete equation by evaluating the integrals in (10). In such a way $F(x,t)$ is constituted by the superposition of two contributions: one corresponding to a damped undistorted wave and the other, with a linear coefficient in a, to a distorted wave [2]:

$$F(x,t)=\frac{1}{2}[\phi(x,t)+\phi(x,-t)]e^{-at}$$
$$+\frac{a}{2}e^{-at}\int_0^t[\phi(x,r)+\phi(x,-r)]\left[I_0(a(t^2-r^2)^{1/2})+\frac{t}{(t^2-r^2)^{1/2}}I_1(a(t^2-r^2)^{1/2})\right]dr \; , \quad (12)$$

where I_0 and I_1 are modified Bessel functions.

In the case of a simple sinusoidal wave $\phi(x,t)=\sin(x-vt)$ we have

$$\tfrac{1}{2}[\phi(x,t)+\phi(x,-t)]=\sin x\,\cos(vt) \quad (13)$$

and the integration in Eq. (12) can be done analytically [2,8]. We obtain

$$F(x,t)=e^{-at}\left[\cos(wt)+\frac{a}{w}\sin(wt)\right]\sin x \; , \quad (14)$$

where $w=(v^2-a^2)^{1/2}$ is an effective velocity [9].

We wish to note that the previous result, Eq. (14) as well as Eq. (13), represents a stationary wave like that we can see in an open-end transmission line [10]. Moreover, by substituting $\sin x$ with the initial amplitude, Eq. (14) also gives the solution of a damped oscillator [2]. In the latter case, when the excitation is represented by a step function, the solution becomes [11]

$$F(t)=1-e^{-at}\left[\cos(wt)+\frac{a}{w}\sin(wt)\right] \; . \quad (15)$$

It is noteworthy that both expressions (14) and (15) resemble the forms adopted in the wavelet analysis [12,13], so that our approach to tunneling could be developed in that framework as well [14]. However, before considering this aspect, we will continue discussing in Sec. II the possibilities offered by electrical networks in simulating quantum tunneling. The implications of wavelet analysis in connection with the tunneling problem will be considered in Sec. III.

II. ELECTRICAL NETWORKS FOR TUNNELING SIMULATION

We recall that the results of a microwave simulation of tunneling are best described by a quantum-mechanical model, suitably translated into the electromagnetic framework, demonstrating that quantum tunneling can be actually simulated with these kinds of experiments [15–17]. An interpretation of this fact can be given along these lines. The analogy between particle motion and electromagnetic wave propagation can be supported on the basis of a similarity in the dispersion relations and of a close correspondence in the wave equations. Feynman, Leighton, and Sands [18], when dealing with waveguides, noted that the dispersion relation for a rectangular waveguide is formally identical to that of a relativistic particle, provided that the proper substitutions are made. Subsequently, a relation has been established between the quantum relativistic motion and the telegrapher's equation which, if analytically continued, results in the Dirac equation [19] and in the Schrödinger relativistic (or Klein-Gordon) equation [4]. Therefore the telegrapher's equation proves to be a suitable tool for studying the propagation of an electromagnetic pulse, which can simulate the motion of a relativistic particle.

Indeed, along these lines, it was possible to derive a simplified model which accounts for the propagation of a signal either above or below the cutoff frequency of the waveguide (classically allowed or forbidden motion, respectively) [3,4]. The starting point is a solution of the telegrapher's equation, similar to Eq. (14),

$$F(x,t)=e^{-at}\left[\sin x\cos(wt)-\frac{v}{w}\cos x\sin(wt)+\frac{a}{w}\sin x\sin(wt)\right], \tag{16}$$

obtained from Eq. (10) in the case of a single progressive wave like $\sin(x-vt)$ in the place of Eq. (13) which, as stated before, represents a stationary wave. Interpreting the effective velocity $w=(v^2-a^2)^{1/2}$ as the beat (or group) velocity, a simple model was obtained according to which the traversal time for a length X is given by

$$\tau=\frac{X}{w_{1,2,3}}, \tag{17}$$

where $w_1\equiv w$ as defined before, while $w_2=(a^2-v^2)^{1/2}$ for $a>v$, and $w_3=(a^2+|v|^2)^{1/2}$ for $v^2<0$ (tunneling case). In the latter two cases ($w_{2,3}$) which arise when w is imaginary, we must also consider imaginary time ($t\to it$) [3]. By identifying w with the group velocity v_g in the waveguide, the transposition of the model into the electromagnetic framework is immediate. The semiclassical delay time, in the absence of dissipation, is given by $\tau=l/|v_g|$, where l is the length of the waveguide, v_g for the TE_{01} mode is $v_g=c[1-(\lambda/2b)^2]^{1/2}$, c is the light velocity, λ is the free-space wavelength, and b is the width of the rectangular waveguide. In the presence of dissipation we consider an effective velocity $\tilde{v}_g$ given by

$$\tilde{v}_g=c\left[1-\left[\frac{\lambda}{2b}\right]^2-\left[\frac{a}{c}\right]^2\right]^{1/2}, \quad \nu>(\nu_0^2+\delta^2)^{1/2} \tag{18}$$

or

$$\tilde{v}_g=c\left[\left[\frac{a}{c}\right]^2-1+\left[\frac{\lambda}{2b}\right]^2\right]^{1/2}, \quad \nu<(\nu_0^2+\delta^2)^{1/2} \tag{19}$$

where $\delta=a/\lambda$. This means that the pure semiclassical model, with a singularity at the cutoff frequency $\nu_0=c/2b$, appears modified so that the singularity is shifted from ν_0 to $\tilde{\nu}_0=(\nu_0^2+\delta^2)^{1/2}$, which can be considered an effective cutoff frequency. The heavy continuous line in Fig. 1 represents the semiclassical model (τ_s) here described which, because of dissipation, shows a shift in the cutoff frequency with respect to the nominal one. In the same figure we report data of delay time (solid circles) and the curve of τ_ϕ (phase-time model), as well as data for τ_z (open circles) and the relative theoretical curve. Without entering now into the details of the several theoretical models, we can say schematically that τ_ϕ represents the real part, while τ_z represents the imaginary part of the traversal time, considered as a complex quantity [4]. Upon inspection of Fig. 1, what clearly emerges is the good agreement of the experimental results, obtained with microwave simulation, with the corresponding theoretical curves as deduced from quantum-mechanical models. As for the prediction of the modified semiclassical model, the relative curve τ_s seems to be in agreement with the absolute value of the complex traversal time $(\tau_\phi^2+\tau_z^2)^{1/2}$ rather than with each (real or imaginary) component. So, on this basis, we may conclude that the path-integral treatment of the telegrapher's equation, for the presence of dissipation, makes the semiclassical model a suitable candidate for interpreting experimental data [3]. Nevertheless, several aspects remain unexplained, such as the fact that the dependence of the delay time versus the barrier length is far from linear, while the semiclassical model, as seen in Eq. (17), essentially shows a linear dependence [4]. The nonlinearity of the delay versus length [in the classically allowed region the dependence is more than linear as predicted by Eq. (9), while in the tunneling region we find an opposite behavior] implies that, for sufficiently long length, the

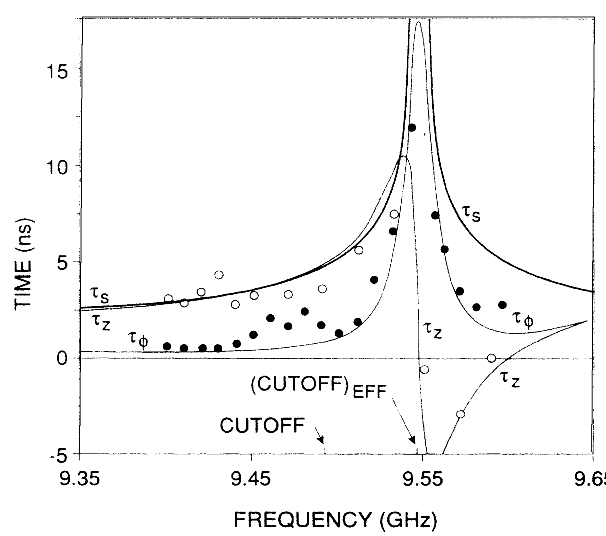

FIG. 1. Data of delay time for $l=15$ cm (solid circles) are to be compared with the curve of τ_ϕ as well as data of τ_z (open circles) with the relative curve. The heavy line τ_s represents the modified semiclassical model resulting from the telegrapher's equation with $a=0.1c$, which implies a shift in the cutoff frequency from $\nu_0\simeq 9.49$ GHz to $\tilde{\nu}_0\cong 9.54$ GHz (after Ref. [3]).

tunneling can result in a superluminal motion [20].

As seen previously the effective (imaginary) velocity in tunneling processes is increased by dissipation and can actually overcome the light velocity c, contrary to the classically allowed motion whereby the effective velocity has c as upper bound [compare Eqs. (18) and (19)]. An extension of the analysis for comparing results of the traversal—or delay—time relative to a beat-envelope signal with those as deduced from the distribution function of the randomized time [$g(r,t)$ in Eq. (10), and its analytical continuation in imaginary time] allowed us to hypothesize that in tunneling processes the traversal time is given by

$$t=\frac{1}{2a}(1-e^{-2aX/v}) , \tag{20}$$

which is just the inverse function of Eq. (9) [7]. However, this fact does not seem sufficient for explaining the extreme shortening of the real delay time below the cutoff (solid circles in Fig. 1). In other words, since there is no phase variation of the wave inside the barrier (evanescent waves), the pulse transit through the barrier itself seems nearly instantaneous. Consequently, anomalous pulse delays originate in microwave propagation [21].

On the basis of these considerations, it seems worthwhile to reconsider Eq. (14) relative to a stationary wave, rather than Eq. (16) relative to a progressive wave. As clearly shown by Eq. (14), and even more by Eq. (13), we are concerned with a superposition of two waves traveling in opposite directions, one towards the positive x coordinate and the other towards the negative, achievable with a section of line with open end (reflection coefficient equal to 1). It is well known that a section of line of length l with a unitary reflection coefficient is equivalent to an electric network constituted by resonant or antiresonant circuits, whose resonant frequencies are given by [22,23]

$$\nu_n=\left[n+\frac{1}{2}\right]\frac{1}{2l\sqrt{LC}} \quad (n=0,1,2,\ldots) , \tag{21}$$

L and C being the inductance and the capacitance per unit length, respectively. If we limit ourselves to the neighborhood of a resonant frequency, for instance, ν_0, it then appears natural to consider the behavior of a single resonant circuit. This way of looking at tunneling simulation, with a circuit where the dimension of length is lost, can be regarded as a zero-dimensional system. This agrees with the adoption of Eq. (14), where the spatial dependence (sin$x$) is merely a factor amplitude.

Now we must analyze the temporal response of these kinds of circuits. This can be done either by a pulse or by a stationary analysis. Let us consider, for example, the circuit of Fig. 2 which represents a typical stage of a shunt-compensated amplifier. The response to a step signal can be written in our terminology as [24]

$$\frac{e_0}{g_m R\, e_i}=1-e^{-a_1 t}\left[\cos(w_1 t)+\frac{a_1}{w_1}\sin(w_1 t)\right.$$
$$\left.-\frac{v^2}{2a_1 w_1}\sin(w_1 t)\right] , \tag{22}$$

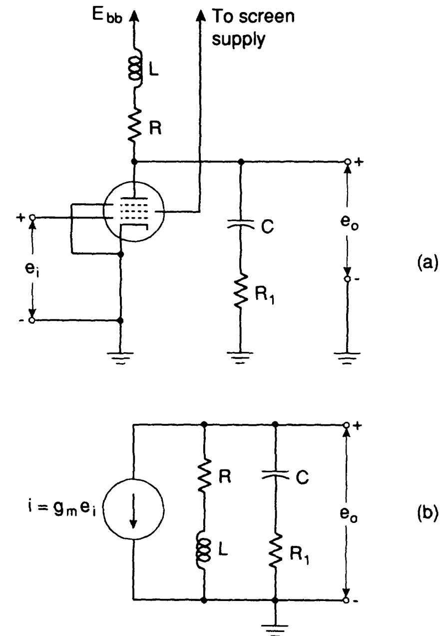

FIG. 2. A shunt-compensated stage (a) and its equivalent circuit (b) (after Ref. [24]). A modified version of the circuit is obtained with inclusion of the resistance R_1 in the capacitive branch.

where $a_1=R/2L$, $w_1=(v^2-a_1^2)^{1/2}$, $v=1/\sqrt{LC}$, with R, L, and C the parameters of the lumped circuit ($R_1\equiv 0$). This expression, which closely resembles Eq. (15), can be discussed in connection to the several cases of w (real and imaginary). On this basis, we will recover a model which resembles the one described in Ref. [27]. However, we prefer to proceed first by a stationary analysis of the circuit, reconsidering the step response later. Now, we report the results relative to a slightly modified version of the same circuit, with the inclusion of a resistive element R_1 in the capacitive branch as shown in Fig. 2. By a standard analysis we find that the amplitude response in a stationary regime is given by

$$\left|\frac{e_0}{e_i}\right|=g_m R\left[\frac{\left[1-\frac{k}{K}x^2\right]^2+x^2\left[k+\frac{1}{K}\right]^2}{(1-x^2)^2+x^2(k+K)^2}\right]^{1/2} \tag{23}$$

and the dephasing by (θ is taken to be positive when the output signal lags the input)

$$\theta=\tan^{-1}\left[\frac{x^3(1-k^2)-x(1-K^2)}{kx^4+kK(k+K)x^2+K}\right] , \tag{24}$$

where $K=R\sqrt{C/L}$, $k=R_1\sqrt{C/L}$, $x=\omega/\omega_0$, $\omega_0=1/\sqrt{LC}$ (for $R_1=0$ we have $k=0$ and we recover the expressions reported in Ref.[24]). The phase delay is given by $D_{\rm ph}=\theta/\omega$, while the group delay is given by

$$D_{\rm gr}=\frac{d\theta}{d\omega}=D_{\rm ph}+\omega\frac{dD_{\rm ph}}{d\omega}=D_{\rm ph}+\frac{d}{d(\ln\omega)}D_{\rm ph} . \tag{25}$$

The results relative to a typical case are shown in Fig. 3.

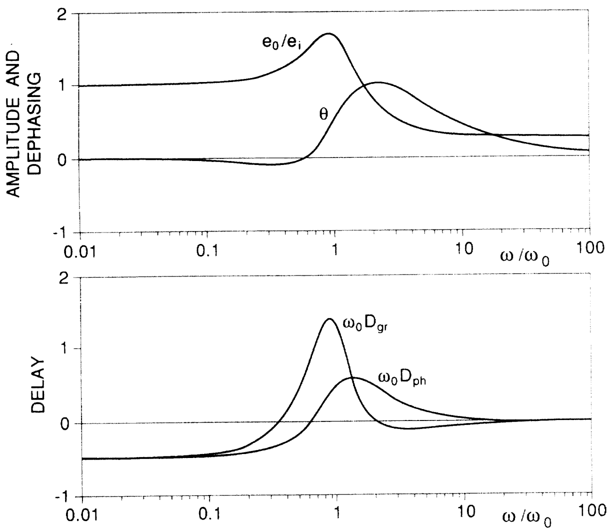

FIG. 3. Response of the modified circuit of Fig. 2 in a stationary regime as a deduced from Eqs. (23)–(25) for $K=0.8$, $k=0.2$, $g_mR=1$.

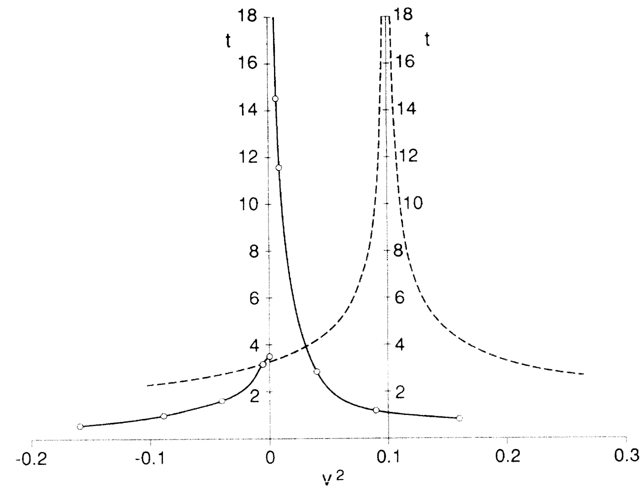

FIG. 4. Delay time (in arbitrary units) deduced from Eqs. (22) and (29) for different values of v^2 and $a_1=0.1$. The dashed line represents the modified semiclassical model of Eq. (17) whose peak is situated at $\bar{a}^2=a^2+a_1^2=0.09+0.01=0.1$ and $X=1$.

We note that the group delay shows an interesting shape which closely resembles the curve of τ_ϕ in Fig. 1, and we can therefore argue that similar results can be obtained using this kind of device.

Returning now to the case of a step-function signal, Eq. (22), we wish to establish a closer correspondence with the quantum tunneling situation. For this purpose, we report the salient features of a treatment of the signal according to the Sommerfeld-Brillouin procedure for a dispersive medium [25]. The propagation of the pulse, representative of a particle of mass m and energy $E=\hbar\omega_0$ traveling in the x direction and subjected to a potential V_0, is described by a contour integral in the complex plane of ω as [26]

$$\psi(x,t)=\frac{1}{2\pi i}\int_\gamma \frac{\exp[-i(\omega t-kx)]}{\omega-\omega_0}d\omega , \tag{26}$$

where γ is a closed path including ω_0 and the momentum $k=\hbar^{-1}[2m(E-V_0)]^{1/2}$ becomes imaginary for $E<V_0$. The pulse is initially described by a function of the type

$$f(t)=\Theta(t)\exp(-i\omega_0 t) , \tag{27}$$

where $\Theta(t)$ is the Heaviside step function. By introducing a new variable with the position $z^2=\omega-(V_0/\hbar)$, Eq. (26) becomes

$$\psi(\xi,t)=\frac{1}{2\pi i}\exp\left[-\frac{iV_0t}{\hbar}\right] \times\int_\Gamma \frac{\exp[i(z\xi-z^2t)]}{z^2-\Omega}2z\,dz , \tag{28}$$

where $\xi=x(2m/\hbar)^{1/2}$, $\Omega=\hbar^{-1}(E-V_0)$, and the contour of integration Γ in the z complex plane is taken according to the steepest-descent criterion. Without entering into a description of the analysis, we limit ourselves to summarize the salient features of the results [27]. The *envelope* of the wave function $\psi(x,t)$ turns out to be shaped in a similar way to Eq. (22) for $\Omega>0$ (classically allowed region) and for $\Omega<0$ (tunneling region) to the corresponding expression of Eq. (22) for imaginary w, that is,

$$\frac{e_0}{g_mR\,e_i}=1-e^{-a_1t}\left[\cosh(w_{2,3}t)+\frac{a_1}{w_{2,3}}\sinh(w_{2,3}t) -\frac{v^2}{2a_1w_{2,3}}\sinh(w_{2,3}t)\right] , \tag{29}$$

where $w_2=\sqrt{a_1^2-v^2}$, for $a_1>v$, and $w_3=\sqrt{a_1^2+|v|^2}$, for $v^2<0$. By Eqs. (22) and (29) we can evaluate the delay time of the signal, taken as the time required to arrive at one-half of its maximum amplitude. The results in a typical case are shown in Fig. 4, and they closely resemble the results of Ref. [27], although the dependence on the length is lost.

III. WAVELET ANALYSIS OF TUNNELING

The results of the preceding section can be assumed as a starting point for analyzing the propagation of a steplike signal according to the telegrapher's equation (note that in doing such we recover the dependence on the length). Namely, we assume that the signal computed according to Eqs. (22) and (29) can be considered a solution $\phi(x,t)$ of the wave equation (1) without dissipation. The effect of the line dissipation, parameter a in Eq. (1), could be considered by working out Eq. (10). This, however, cannot be easily done directly in an analytical way: it is at this point that we invoke the concept of wavelet for solving our problem. As anticipated, analytical forms like Eqs. (14) and (15), but also Eqs. (22) and (29), can be considered as wavelets, or more precisely, coherent states, depending on two parameters a and b, of the type [12]

$$G_{a,b}(x)=G(x-b)e^{iax} , \tag{30}$$

where $G(x)$ is a window function—generally a Gaussian function but any other physically acceptable function can be used. The parameters a and b allow a connection to momentum (or frequency) and position (or time), respectively [28].

A first attempt to find the answer for our system to a steplike function was made considering only the main part of the signal, in the neighborhood of its rise, by developing it in a finite series expansion like [4]

$$F(t)=\frac{1}{2}+\frac{\pi}{2}[A_1\sin(\omega t+\theta_1)+A_3\sin(3\omega t+\theta_3)+\cdots] . \tag{31}$$

The coefficients $A_{1,3,\ldots}$ and $\theta_{1,3,\ldots}$ are determined by a best fit of the signal already evaluated, Eqs. (22) and (29), by identifying ω with w, ω_d (see below) with a, and θ with x. In such a way, we could apply the same procedure described above [see Eq. (16)] for a single sinusoidal wave. Then, by adding again the results as in the original expansion (31), we obtain the signal as modified by propagation in the presence of dissipation. In Fig. 5 we show the results of this treatment in the case of a signal relating to a classically allowed region. As expected, we note that the effect of dissipation is to increase the rise time. When the dissipation exceeds the velocity ($w<a$, that is, $\omega<\omega_d$), $\omega_1=\sqrt{\omega^2-\omega_d^2}$ becomes imaginary and, as in the case of single wave, we considered the analytic continuation of Eq. (16), or (22), in imaginary time with $\omega_2=\sqrt{\omega_d^2-\omega^2}$. Analogously, in the tunneling region we write $\omega_3=\sqrt{\omega_d^2+|\omega|^2}$. In these latter cases ($\omega_{2,3}$), we found that dissipation tends to reduce the arrival time of the signal. So, as in the case of a beat envelope, the effect of the dissipation is to shift the peak of the curve of the delay time towards the high frequencies [4].

Subsequently, in the attempt to avoid entering in imaginary time, we have considered a different kind of expansion of the signal of the type [29]

$$F(t)=\sum_{1}^{p} a_i z_i^t , \tag{32}$$

where a_i and z_i are complex numbers and t is a real quantity. By Eq. (32) we can give a rather accurate description of the signal with a moderate number of terms ($p\approx 3-5$), avoiding the rather artificial criterion of considering an arbitrary small interval around the rise of the signal, adopted in connection with Eq. (31), in order to contain the number of harmonics. In spite of this, an expansion like (32) was not employed for deriving the arrival time because of the difficulty in obtaining an analytical solution of the wave equation with these kinds of functions.

Another attempt can be made searching for the solution of the telegrapher's equation starting directly from a $\phi(x,t)$ derived from Eq. (22), of the type (see Fig. 6)

$$\phi(x,t)\approx\exp\left[-a_1\left|t-\frac{x}{v}\right|\right]\sin(x-vt) , \tag{33}$$

which satisfies the d'Alembert equation, Eq. (1) for $a=0$. The importance of this form lies in the fact that it is "naturally" confined, like the coherent states of Eq. (30), and only two or three terms are sufficient in describing our signals. The difficulty of obtaining a direct analytical

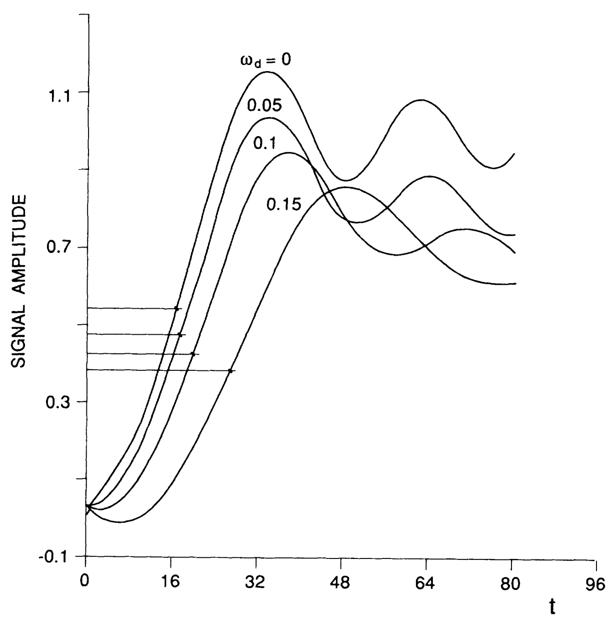

FIG. 5. Signal envelope, in the neighborhood of the rise, for a classically allowed motion. In the absence of dissipation ($\omega_d=0$) the signal is fitted by Eq. (31) with five harmonics. All the sinusoidal components are then modified, according to Eq. (16), for each value of the dissipation parameter ω_d, and then recomposed in order to obtain the signal shape with dissipation. The delay time, taken half way of the maximum amplitude, is indicated.

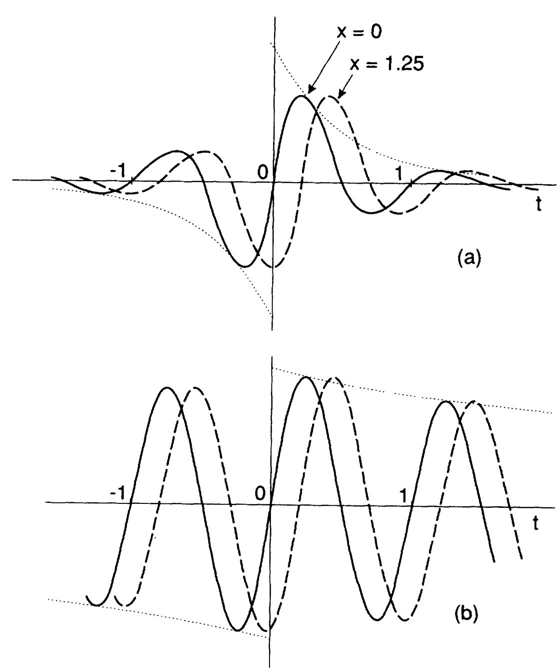

FIG. 6. A "wavelet" $-\phi(x,t)$ [see Eq. (33)] propagating in the x direction represented as a function of the time at two different x positions. Case (a) $a_1=2$, $v=2\pi$, case (b) $a_1=0.2$, $v=2\pi$.

solution of the wave equation, starting with the form (33), can be surmounted as follows.

Let us consider the Laplace transform of the wave equation (1), that is [30],

$$\frac{d^2f}{dx^2}-\frac{1}{v^2}(s^2+2as)f+\frac{s+2a}{v^2}\phi(x,0)+\frac{1}{v^2}\left[\frac{\partial\phi}{\partial t}\right]_{t=0}=0\ , \qquad (34)$$

where $f(x,s)$ is the Laplace transform of $F(x,t)$ which is the solution of Eq. (1). According to Eq. (33) we have

$$\phi(x,0)=e^{-a_1x/v}\sin x\ ,$$

$$\left[\frac{\partial\phi}{\partial t}\right]_{t=0}=e^{-a_1x/v}(a_1\sin x-v\cos x)\ .$$

We can verify that Eq. (34) is satisfied, for $a_1 \ll v$, by [31]

$$f(x,s)\simeq\frac{(s+2a+a_1)\sin x-v\cos x}{s^2+2as+w_1^2}e^{-a_1x/v}\ , \qquad (35)$$

where $w_1=\sqrt{v^2-a_1^2}$. By the inverse Laplace transform of Eq. (35) we find that an approximate solution of Eq. (1), with $\phi(x,t)$ as Eq. (33), is given by

$$F(x,t)\simeq e^{-at}e^{-a_1x/v}\left[\sin x\cos(\tilde{w}t)-\frac{v}{\tilde{w}}\cos x\sin(\tilde{w}t)+\frac{a+a_1}{\tilde{w}}\sin x\sin(\tilde{w}t)\right]\ , \qquad (36)$$

where

$$\tilde{w}=\sqrt{w_1^2-a^2}=\sqrt{v^2-a_1^2-a^2}=\sqrt{v^2-\tilde{a}^2}\ . \qquad (37)$$

On the basis of this last result we can anticipate that the time-delay function versus frequency should be peaked at $v^2=\tilde{a}^2$ rather than at $v^2=a^2$ as in the beat-envelope model [Eq. (17)]. In Fig. 4 this is represented by the dashed curve corresponding to the simplified model of Eq. (17) displaced at $\tilde{a}^2$. For a better description of the delay time versus frequency, we must consider the complete expression (36). We note that Eq. (36) is very similar to Eq. (16), apart from the factor $e^{-a_1x/v}$, the substitution of $w=(v^2-a^2)^{1/2}$ with $\tilde{w}=(w^2-a_1^2)^{1/2}$, and the coefficient of the third term, which contains the parameter a_1.

So, in conclusion, we can recover most of the calculations derived earlier, and in particular a detailed description of the delay time as a function of v^2 as reported in Fig. 1 of Ref.[3]. In that case, the influence of the distorted wave, the third term in Eqs. (16) and (36), is clearly evidenced with an enhancement of the traversal time in the allowed region and a lowering in the tunneling one. In such a way we can get a rough idea of the influence of the parameter a_1 over the delay time, except in the peak region (around the effective cutoff of Fig. 1 or $\tilde{a}^2$ of Fig. 4) since the approximate result of Eq. (36) holds only for $a_1 \ll v$. Further work is required in order to consider a complete description of the input pulse like those of Eqs. (22) and (29) and the relative delay time. This, however, overcomes the purposes of the present work devoted to demonstrating that the wavelet concept represents a more refined approach to the interpretation of electromagnetic simulations of tunneling in connection with the telegrapher's equation.

Note added. After this work was accomplished, we received a report by Nimtz, Enders, and Spieker [32] dealing with tunneling time delay in a microwave cavity. The latter consists of a waveguide section terminated by two subcutoff waveguides as mirrors: in such a way a standing wave regime is set up. The measurements are performed by varying the frequency and by crossing a cavity resonance, the resulting delay shows a marked peak while, out of the resonance, superluminal effects are observed. So, the overall behavior strongly resembles the shape of $\omega_0 D_{gr}$ here shown in Fig. 3, apart from an unessential off zero.

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$$f(x,s)=(A\sin x+B\cos x)e^{-a_1x/v},$$

where the coefficients A and B are determined by solving the following system:

$$-\frac{2a_1}{v}A-\frac{w_1^2+s^2+2as}{v^2}B=\frac{1}{v},$$

$$-\frac{2a_1}{v}B+\frac{w_1^2+s^2+2as}{v^2}A=\frac{s+2a+a_1}{v^2}.$$

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